Review: Derivative Definition - 10/19/16

1 Definition of Derivative

Definition 1.0.1 *The* ***derivative*** *of a function f is*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

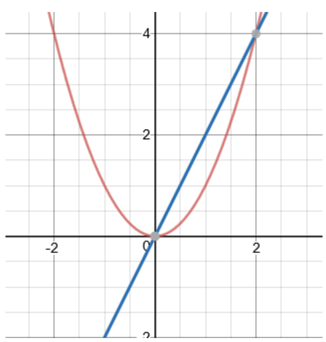
Some alternate notations:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(a) = Df(x) = D_x f(x).$$

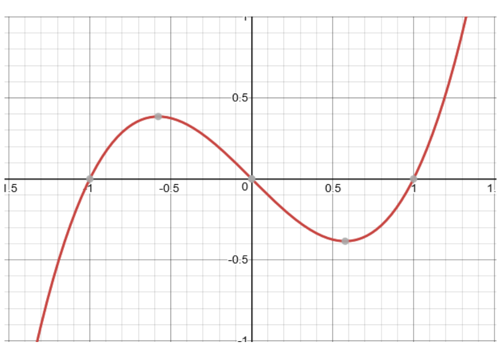
When we want to take the derivative at a point, we can write it as $f'(a)$ or as $\frac{dy}{dx}\Big|_{x=a}$.

2 Sketching the Derivative

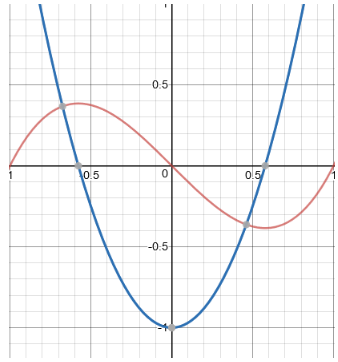
Example 2.0.2 *Let $f(x) = x^2$. What is $\frac{d}{dx} f(x)$? We have $\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$ $= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0} 2x + h = 2x$. Below, the red line is the original function, and the blue is the derivative.*



Example 2.0.3 *Sketch out the derivative of the following graph:*



The slope of the tangent line is positive where the graph is increasing, negative where the graph is decreasing, and zero where it changes direction. If we draw a graph that has those characteristics, we get:



Let's see if what we get algebraically matches up with our picture. The equation of the original function is $f(x) = x^3 - x$. To find the derivative, we take

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h} \\ &= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2 - 1)}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 - 1 = 3x^2 - 1\end{aligned}$$

But this is exactly the graph that we got in the picture!

3 Differentiability

Definition 3.0.4 *A function f is **differentiable at** a if $f'(a)$ exists.*

Theorem 3.0.5 *If f is differentiable at a, then f is continuous at a.*

Example 3.0.6 *Let $f(x) = \sqrt{x}$. Is f continuous at 0? Yes it is: the limit as x goes to 0 is 0. Is f differentiable at 0? We need to check if it has a derivative:*

$$\begin{aligned}f'(0) &= \lim_{h\to 0} \frac{f(0+h) - \sqrt{0}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{h}}{h} \cdot \frac{\sqrt{h}}{\sqrt{h}} \\ &= \lim_{h\to 0} \frac{h}{h\sqrt{h}} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{h}}\end{aligned}$$

The limit of this is infinity. It does not make sense to have a function where if I plug in 0, I get out infinity, so f is not differentiable at 0.

This means that if f is continuous at a, it might be differentiable at a, or it might not be. If f is discontinuous at a, it is definitely NOT differentiable at a.

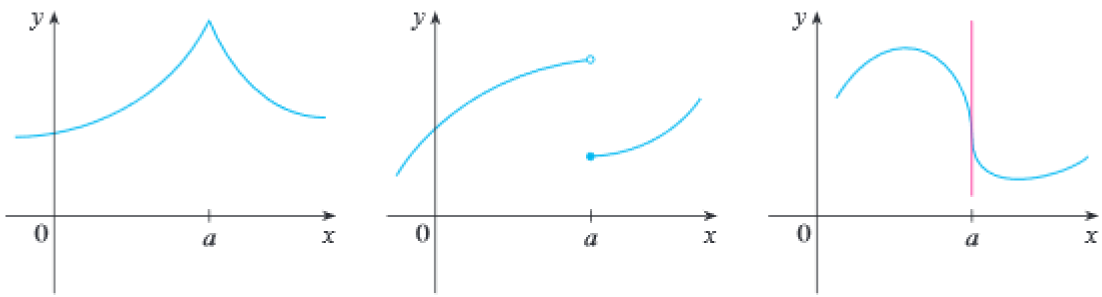
Example 3.0.7 *Let* $g(x) = |x|$. *Is* g *differentiable at 0? We have* $g'(0) = \lim_{h\to 0} \frac{|0+h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$. *We want to know if this exists. Let's take the left and right hand limits. For the right hand limit, our* h *values will be positive as we're approaching 0, so we can take out the absolute value:* $\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$. *For the left hand limit, our* h *values will be negative, so when I take the absolute value, I will have* $-h$*, since the minus will cancel the fact that* h *is negative to give me a positive number:* $\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1$. *Do the left and right hand limits equal one another? No, so the limit does not exist. That means that* g *is not differentiable at 0.*

In summary, we've seen three ways in which a function f is not differentiable at a point a:

(a) f has a kink at a. (this is where the limit does not exist).

(b) f is discontinuous at a.

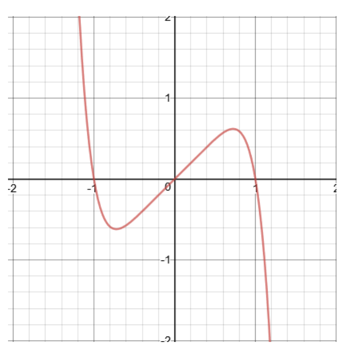
(c) f has a vertical tangent line at a. (this happened with $\sqrt{x}$)



(a) A corner (b) A discontinuity (c) A vertical tangent

Practice Problems

1. Let $f(x) = \frac{3}{x}$. What is $f'(x)$?
2. Let $g(x) = x^2 + 2$. Is $g(x)$ differentiable?
3. Let $k(x) = \frac{1}{x^2}$. If $k(x)$ differentiable at 1? At 0?
4. Sketch the derivative of the following graph:



Solutions

1.

$$f'(x) = \lim_{h\to 0} \frac{\frac{3}{x+h} - \frac{3}{x}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{3x-3(x+h)}{x(x+h)}}{h}$$
$$= \lim_{h\to 0} \frac{-3h}{h(x(x+h))}$$
$$\lim_{h\to 0} \frac{-3}{x^2+xh} = \frac{-3}{x^2}$$

2. To do this, we need to find the derivative of g to see if it exists:

$$g'(x) = \lim_{h\to 0} \frac{(x+h)^2+2-(x^2+2)}{h}$$
$$= \lim_{h\to 0} \frac{x^2+2xh+h^2+2-x^2-2}{h}$$
$$= \lim_{h\to 0} \frac{h(2x+h)}{h}$$
$$= \lim_{h\to 0} 2x+h = 2x$$

Since $2x$ exists for all x, then the derivative of g exists everywhere, so g is differentiable.

3. To see if k is differentiable at 1, we have to see if $k'(1)$ exists:

$$k'(1) = \lim_{h\to 0} \frac{\frac{1}{(1+h)^2} - 1}{h}$$
$$= \lim_{h\to 0} \frac{1-(1+h)^2}{h(1+h)^2}$$
$$= \lim_{h\to 0} \frac{1-1-2h-h^2}{h(1+h)^2}$$
$$= \lim_{h\to 0} \frac{h(-2-h)}{h(1+h)^2}$$
$$= \lim_{h\to 0} \frac{-2-h}{(1+h)^2} = -2$$

Since the derivative exists, then k is differentiable at 1. Not that $k(x)$ is not continuous at 0 (since 0 is not in the domain), so it is not differentiable at 0.

4. The red line is the function and the blue line is the derivative.